

Quantum Speed Limit For Mixed States Using Experimentally Realizable Metric

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The minimal time required for a system to evolve between two different states is an important notion for developing ultra-speed quantum computer and communication channel. Here, we introduce a new metric for non-degenerate density operator evolving along unitary orbit and show that this is experimentally realizable operation dependent metric on quantum state space. Using this metric, we obtain the geometric uncertainty relation that leads to a new quantum speed limit. Furthermore, we argue that this gives a tighter bound for the evolution time compared to any other bound. We also obtain a Levitin kind of bound for mixed states. We propose how to measure this new distance and speed limit in quantum interferometry. Finally, the lower bound for the evolution time of a quantum system is studied for any completely positive trace preserving map using this metric.

Introduction.— Various attempts are being made in the laboratory to implement quantum gates, which are the building blocks of a quantum computer. Performance of a quantum computer is determined by how fast one can apply these logic gates so as to drive the initial state to a final state. Then, the natural question that arises is: can a quantum state evolve arbitrarily faster? It turns out that quantum mechanics limits the evolution speed of any quantum system. In quantum information, study of these limits has found several applications over the years. Some of these include, but not limited, to quantum metrology, quantum chemical dynamics, quantum control and quantum computation.

Extensive amount of work has already been done on the subject “minimum time required to reach a target state” since the appearance of first major result by Mandelstam and Tamm [1]. However, the notion of quantum speed or speed of transportation of quantum state was first introduced by the Anandan-Aharonov using the Fubini-Study metric [2] and subsequently, the same notion was defined by Pati [3] using the Riemannian metric [4]. It was found that the speed of a quantum state on the projective Hilbert space is proportional to the fluctuation in the Hamiltonian of the system. Using the concept of Fubini-Study metric on the projective Hilbert space, a geometric meaning is given to the probabilities of a two-state system [5]. Furthermore, it was shown that the quantum speed is directly related to the super current in the Josephson junction [6]. In the last two decades, there have been various attempts made in understanding the geometric aspects of quantum evolution for pure as well for mixed states [7–45]. The quantum speed limit for the driven [45] and the non-Markovian [46] quantum systems was introduced using the notion of Bures metric [47]. Very recently, the quantum speed limit in the case of open quantum system [48] was introduced using the notion of relative purity [49]. In an interesting twist, it has been shown that the quantum speed limit for multipartite system is bounded by the generalised geometric measure of entanglement [50].

In this letter, we introduce a new metric, which can be measured experimentally in interference of mixed states. We show that using this metric, it is possible to define a new lower limit

for the evolution time of any system described by mixed state undergoing unitary evolution. We derive the quantum speed limit using the geometric uncertainty relation based on this new metric. We also obtain a Levitin kind of bound for mixed states using our approach. We show that this bound for the evolution time of a quantum system is tighter than any other existing bounds for unitary evolutions. Most importantly, we propose an experiment to measure this new distance in the interference of mixed states. We argue that the visibility in quantum interference is a direct measure of distance for mixed quantum states. Finally, we generalize this idea for the case of completely positive trace preserving evolutions and get a new lower bound for the evolution time using this metric.

Metric along unitary orbit.— Let \mathcal{H} denotes a finite-dimensional Hilbert space and $\mathcal{L}(\mathcal{H})$ is the set of linear operators on \mathcal{H} . A density operator ρ is a Hermitian, positive and trace class operator that satisfies $\rho \geq 0$ and $\text{Tr}(\rho) = 1$. Let ρ be a non-degenerate density operator with spectral decomposition $\rho = \sum_k \lambda_k |k\rangle\langle k|$, where λ_k 's are the eigenvalues and $\{|k\rangle\}$'s are the eigenstates. We consider a system at time t_1 in a state ρ_1 . It evolves under a unitary evolution and at time t_2 , the state becomes $\rho_2 = U(t_2, t_1)\rho_1 U^\dagger(t_2, t_1)$. Any two density operators that are connected by a unitary transformation will give a unitary orbit. If $U(N)$ denotes the set of $N \times N$ unitary matrices on \mathcal{H}^N , then for a given density operator ρ , the unitary orbit is defined by $\rho' = \{U\rho U^\dagger : U \in U(N)\}$. The most important notion that has resulted from the study of interference of mixed quantum states is the concept of the relative phase between ρ_1 and ρ_2 and the notion of visibility in the interference pattern. The relative phase is defined by [51]

$$\Phi(t_2, t_1) = \text{ArgTr}[\rho_1 U(t_2, t_1)] \quad (1)$$

and the visibility is defined by

$$V = |\text{Tr}[\rho_1 U(t_2, t_1)]|. \quad (2)$$

The quantity $\text{Tr}[\rho_1 U(t_2, t_1)]$ represents the probability amplitude between ρ_1 and ρ_2 , which are unitarily connected. Therefore, for the unitary orbit $|\text{Tr}(\rho_1 U(t_2, t_1))|^2$ represents

the transition probability between ρ_1 and ρ_2 . Note that if $\rho_1 = |\psi_1\rangle\langle\psi_1|$ is a pure state and $|\psi_1\rangle = |\psi(t_1)\rangle \rightarrow |\psi_2\rangle = |\psi(t_2)\rangle = U(t_2, t_1)|\psi(t_1)\rangle$, then $|\text{Tr}[\rho_1 U(t_2, t_1)]|^2 = |\langle\psi(t_1)|\psi(t_2)\rangle|^2$, which is nothing but the fidelity between two pure states.

All the existing metrics on the quantum state space give rise to the distance between two states independent of the operation. Here, we define a new distance between two unitarily connected states of a quantum system. This distance not only depends on the states but also depends on the operation under which the evolution occurs. Whether a state of a system will evolve to another state depends on the Hamiltonian which in turn fixes the unitary orbit. Let the mixed state traces out an open unitary curve $\Gamma : t \in [t_1, t_2] \rightarrow \rho(t)$ in the space of density operators with end points ρ_1 and ρ_2 . If the unitary orbit connects the state ρ_1 at time t_1 to ρ_2 at time t_2 , then the distance between them is defined by

$$D(\rho_1, \rho_2)^2 := 4(1 - |\text{Tr}[\rho_1 U(t_2, t_1)]|^2), \quad (3)$$

which also depends on the orbit. We will show that it is indeed a metric, i.e., it satisfies the conditions (i) $D(\rho_1, \rho_2) \geq 0$ (non-negative) and 0 if and only if there is no evolution i.e., $[\rho_1, U] = 0$, (ii) symmetric, i.e., $D(\rho_1, \rho_2) = D(\rho_2, \rho_1)$ and (iii) the triangle inequality, i.e., for any three density operators ρ_1, ρ_2 and ρ_3 , which are unitarily connected, we have $D(\rho_1, \rho_3) \leq D(\rho_1, \rho_2) + D(\rho_2, \rho_3)$.

We know that for any operator A and a unitary operator U , $|\text{Tr}(AU)| \leq \text{Tr}|A|$ with equality for $U = V^\dagger$, where $A = |A|V$ is the polar decomposition of A [55]. Considering $A = \rho = |\rho|$, we get $|\text{Tr}[\rho_1 U(t_2, t_1)]| \leq 1$. It can also be shown that $D(\rho_1, \rho_2) = 0$ if and only if $[\rho_1, U] = 0$ [56], which proves the condition (i). To prove the condition (ii), we show that the quantity $|\text{Tr}[\rho_1 U(t_2, t_1)]|$ is symmetric with respect to the initial and the final states. In particular, we have

$$\begin{aligned} |\text{Tr}[\rho_1 U(t_2, t_1)]| &= |\text{Tr}[U^\dagger(t_2, t_1)\rho_2 U(t_2, t_1)]| \\ &= |\text{Tr}[\rho_2 U(t_2, t_1)]| = |\text{Tr}[\rho_2 U^\dagger(t_2, t_1)]| \end{aligned} \quad (4)$$

and hence (ii). To see that the new distance satisfies the triangle inequality, consider its purification. Let $\rho_A(t_1)$ and $\rho_A(t_2)$ are two unitarily connected mixed states of a quantum system A . If we consider the purification of $\rho_A(t_1)$, then we have $\rho_A(t_1) = \text{Tr}_B[|\Psi_{AB}(t_1)\rangle\langle\Psi_{AB}(t_1)|]$, where $|\Psi_{AB}(t_1)\rangle = (\sqrt{\rho_A(t_1)}V_A \otimes V_B)|\alpha\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, V_A, V_B are local unitary operators and $|\alpha\rangle = \sum_i |i^A i^B\rangle$. The evolution of $\rho_A(t_1)$ under $U_A(t_2, t_1)$ is equivalent to the evolution of the pure state $|\Psi_{AB}(t_1)\rangle$ under $U_A(t_2, t_1) \otimes I_B$ in the extended Hilbert space. Thus, in the extended Hilbert space, we have $|\Psi_{AB}(t_1)\rangle \rightarrow |\Psi_{AB}(t_2)\rangle = U_A(t_2, t_1) \otimes I_B |\Psi_{AB}(t_1)\rangle$. Now consider the transition amplitude between two states $\langle\Psi_{AB}(t_1)|\Psi_{AB}(t_2)\rangle$, which is given by

$$\begin{aligned} &\langle\Psi_{AB}(t_2)|\Psi_{AB}(t_1)\rangle \\ &= \text{Tr}(U_A \sqrt{\rho_A(t_1)} V_A \otimes V_B |\alpha\rangle\langle\alpha| V_A^\dagger \sqrt{\rho_A(t_1)} \otimes V_B^\dagger) \\ &= \text{Tr}(|\alpha\rangle\langle\alpha| V_A^\dagger \sqrt{\rho_A(t_1)} \otimes V_B^\dagger U_A \sqrt{\rho_A(t_1)} V_A \otimes V_B) \end{aligned}$$

$$\begin{aligned} &= \sum_{i,j} \delta_{ij} \cdot \langle j^A | U_A \sqrt{\rho_A(t_1)} V_A V_A^\dagger \sqrt{\rho_A(t_1)} | i^A \rangle \\ &= \text{Tr}[\rho_A(t_1) U_A(t_2, t_1)] \end{aligned} \quad (5)$$

This simply says that the expectation value of a unitary operator $U_A(t_2, t_1)$ in a mixed state is equivalent to the inner product between two pure states in the enlarged Hilbert space. This inner product is nothing but the overlap between the purified version of the mixed state with its unitarily evolved counterpart. Since, in the extended Hilbert space the purified version of the metric satisfies the triangle inequality, hence the condition (iii) holds also for mixed states. Thus, if we evolve a state ρ_1 under a unitary operator $U(t_2, t_1)$ and get ρ_2 , the metric defined in terms of this expectation value of the unitary operator $U(t_2, t_1)$ in the ρ_1 expresses the distance between ρ_1 and ρ_2 along the unitary evolution path. If ρ_1 and ρ_2 are two pure states, which are unitarily connected then our new metric is the Fubini-Study metric [2, 3, 54] on the projective Hilbert space $\mathbf{CP}(\mathcal{H})$.

Now, imagine that two density operators differ from each other in time by an infinitesimal amount, i.e., $\rho(t_1) = \rho(t) = \sum_k \lambda_k |k\rangle\langle k|$ and $\rho(t_2) = \rho(t + dt)$. Then, the infinitesimal distance between them is given by

$$dD^2 = 4(1 - |\text{Tr}[\rho(t)U(dt)]|^2). \quad (6)$$

If we use the time evolution equation for the unitary operator, then we have

$$\begin{aligned} dD^2 &= \frac{4}{\hbar^2} [\text{Tr}(\rho(t)H^2) - [\text{Tr}(\rho(t)H)]^2] dt^2 \\ &= \frac{4}{\hbar^2} [\sum_k \lambda_k \langle k|\dot{k}\rangle - (i \sum_k \lambda_k \langle k|\dot{k}\rangle)^2] dt^2. \end{aligned} \quad (7)$$

Therefore, the total distance travelled by the density operator along the unitary orbit is given by

$$D = \frac{2}{\hbar} \int (\Delta H)_\rho dt, \quad (8)$$

where $(\Delta H)_\rho$ is the uncertainty in the Hamiltonian of the system in the state ρ and is defined as $(\Delta H)_\rho^2 = [\text{Tr}(\rho(t)H^2) - [\text{Tr}(\rho(t)H)]^2]$. Thus, it is necessary and sufficient to have non-zero ΔH for quantum system to evolve in time.

Quantum speed limit.— We consider a system A with mixed state $\rho_A(0)$ at time $t=0$, which evolves under a unitary operator $U_A(T)$. We define the Bargmann angle for two unitarily connected mixed states $\rho_A(0)$ at time $t=0$ and $\rho_A(T)$ at time $t=T$ as

$$|\text{Tr}[\rho_A(0)U_A(T)]| = \cos \frac{s_0}{2}, \quad (9)$$

such that $s_0 \in [0, \pi]$. Note that s_0 can also be expressed in terms of the purifications of the states $\rho_A(0)$ and $\rho_A(T)$ in the extended Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, i.e., $|\text{Tr}[\rho_A(0)U_A(T)]| = |\langle\Psi_{AB}(0)|\Psi_{AB}(T)\rangle| = \cos \frac{s_0}{2}$, where $|\Psi_{AB}(T)\rangle = U_A(T, 0) \otimes I_B |\Psi_{AB}(0)\rangle$. The inequality in the

extended Hilbert space now becomes a property of the projective Hilbert space, i.e., $D \geq s_0$ [1, 2]. Using the inequality $D \geq s_0$ and the fact that the system Hamiltonian H is time independent, we get the time limit of the evolution as

$$T \geq \frac{\hbar s_0}{2\overline{\Delta H}}. \quad (10)$$

This same idea can be extended for the quantum system with time dependent Hamiltonian. The speed limit in this case is given by

$$T \geq \frac{\hbar s_0}{2\overline{\Delta H}}, \quad (11)$$

where $\overline{\Delta H} = (\frac{1}{T} \int_0^T \Delta H dt)$ is the time averaged energy uncertainty of the quantum system. This may be considered as generalization of the Anandan-Aharonov geometric uncertainty relation for the mixed states. This bound is better and tighter than the bound given in [27, 45] and reduces to the time limit given by Anandan and Aharaonov [2] for pure states. There can be some states called intelligent states and some optimal Hamiltonians for which the equality may hold. But in general, it is difficult to find such intelligent states [23, 25].

To see that Eq. (10) indeed gives a tighter bound, consider the following. We suppose that a system in a mixed state ρ_A evolves to ρ'_A under $U_A(t)$. Let S and S' are the sets of purifications of ρ_A and ρ'_A respectively. In [27, 45], time bound was given in terms of Bures metric [47], i.e., $\min_{|\Psi_{AE}\rangle, |\Phi_{AE}\rangle} 2 \cos^{-1} |\langle \Psi_{AE} | \Phi_{AE} \rangle|$ [55], such that $|\Psi_{AE}\rangle \in S$ and $|\Phi_{AE}\rangle \in S'$. But in Eq. (10), the time bound is tighter than that given in [27, 45] in the sense that here the bound is in terms of s_0 , i.e., $s_0 = 2 \cos^{-1} |\langle \Psi_{AE} | \Phi_{AE} \rangle|$, such that $|\Phi_{AE}\rangle = U_A \otimes I_E |\Psi_{AE}\rangle$ and hence, s_0 is always greater than or equals to the Bures angle [47] defined as $2 \cos^{-1} [\text{Tr} \sqrt{\rho_A^{\frac{1}{2}} \rho'_A \rho_A^{\frac{1}{2}}}]$.

Using our formalism, we can also derive Margolus and Levitin kind of time bound [24] for the mixed states. For this we consider the system A with a mixed state $\rho(0)$ at time $t=0$. Let $\rho(0) = \sum_k \lambda_k |k\rangle \langle k|$ be the spectral decomposition of $\rho(0)$ and it evolves under a unitary operator $U(T)$ to a state $\rho(T)$. In this case, we have

$$\begin{aligned} \text{Tr}[\rho(0)U(T)] &= \sum_k \lambda_k \langle k | U(T) | k \rangle \\ &= \sum_n p_n \left(\cos \frac{E_n T}{\hbar} + i \sin \frac{E_n T}{\hbar} \right), \end{aligned} \quad (12)$$

where we have used $|k\rangle = \sum_n c_n^{(k)} |\psi_n\rangle$, and $|\psi_n\rangle$'s are eigenstates of the Hamiltonian H with $H|\psi_n\rangle = E_n |\psi_n\rangle$, and $p_n = \sum_k \lambda_k |c_n^{(k)}|^2$. Using the inequality $\cos x \geq 1 - \frac{2}{\pi}(x + \sin x)$ for $x \geq 0$, i.e., for positive semi-definite Hamiltonian, we get

$$\begin{aligned} \text{Re}[\text{Tr}[\rho(0)U(T)]] &= \sum_n p_n \cos \frac{E_n T}{\hbar} \\ &\geq \left[1 - \frac{2}{\hbar\pi} T \langle H \rangle - \frac{2}{\pi} \sum_n p_n \sin \frac{E_n T}{\hbar} \right] \end{aligned} \quad (13)$$

Then, from Eq. (13), we have

$$T \geq \frac{\pi\hbar}{2\langle H \rangle} [1 - R - \frac{2}{\pi} I], \quad (14)$$

where R and I are real and imaginary parts of $\text{Tr}[\rho(0)U(T)]$ and they can be positive as well as negative. Note that when R and I are negative, this can give a tighter bound. This new time bound for mixed states evolving under unitary evolution with non-negative Hamiltonian reduces to $\frac{\hbar}{4\langle H \rangle}$, i.e., Margolus and Levitin [24] bound in the case of evolution from one pure state to its orthogonal state. Therefore, the time limit of the evolution under unitary operation with Hamiltonian H becomes

$$T \geq \begin{cases} \frac{\hbar s_0}{2\overline{\Delta H}} & \text{if } H \text{ is not positive semi-definite} \\ \max\left\{ \frac{s_0\hbar}{2\overline{\Delta H}}, \frac{\pi\hbar}{2\langle H \rangle} \left(1 - \frac{2}{\pi} I - R\right) \right\} & \text{if } H \text{ is positive semi-definite} \end{cases} \quad (15)$$

In the following, we have taken an example in the two dimensional state space and shown that the inequality is indeed satisfied by the quantum system.

Example of speed limit for unitary evolution.— We consider a general single qubit state $\rho(0) = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})$, such that $|\vec{r}|^2 \leq 1$. Let it evolves under a general unitary operator U , i.e., $\rho(0) \rightarrow \rho(T) = U(T)\rho(0)U^\dagger(T)$, where $U = e^{i\frac{a}{\hbar}(\hat{n} \cdot \vec{\sigma} + \alpha I)}$, $a = \omega T$ and the Hamiltonian $H = \omega(\hat{n} \cdot \vec{\sigma} + \alpha I)$ ($\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices and \hat{n} is a unit vector). This Hamiltonian H becomes positive semi-definite for $\alpha \geq 1$. It is easy to show, that for $\alpha = 1$, $s_0 = 2 \cos^{-1} \sqrt{[\cos^2 \frac{a}{\hbar} - (\hat{n} \cdot \vec{r}) \sin^2 \frac{a}{\hbar}]^2 + \cos^2 \frac{a}{\hbar} \sin^2 \frac{a}{\hbar} (1 + \hat{n} \cdot \vec{r})^2}$, $\Delta H = \omega \sqrt{1 - (\hat{n} \cdot \vec{r})^2}$, $R = [\cos^2 \frac{a}{\hbar} - (\hat{n} \cdot \vec{r}) \sin^2 \frac{a}{\hbar}]$, $I = \cos \frac{a}{\hbar} \sin \frac{a}{\hbar} (1 + \hat{n} \cdot \vec{r})$ and $\langle H \rangle = \omega(1 + \vec{r} \cdot \hat{n})$. Using the inequality (15), for $\alpha = 1$, we get

$$T \geq \max \left\{ \frac{\hbar \cos^{-1} \sqrt{[\cos^2 \frac{a}{\hbar} - (\hat{n} \cdot \vec{r}) \sin^2 \frac{a}{\hbar}]^2 + \cos^2 \frac{a}{\hbar} \sin^2 \frac{a}{\hbar} (1 + \hat{n} \cdot \vec{r})^2}}{\omega \sqrt{1 - (\hat{n} \cdot \vec{r})^2}}, \frac{\frac{\pi\hbar}{2\langle H \rangle} [1 - \cos^2 \frac{a}{\hbar} + (\hat{n} \cdot \vec{r}) \sin^2 \frac{a}{\hbar}]}{-\frac{2}{\pi} \cos \frac{a}{\hbar} \sin \frac{a}{\hbar} (1 + \hat{n} \cdot \vec{r})} \right\}, \quad (16)$$

where $(\hat{n} \cdot \vec{r})^2 \in [0, 1]$. For $\hbar = \omega = 1$ and $a = \pi/2$, the initial state evolves to $\rho(T) = \frac{1}{2}(I + \vec{r}' \cdot \vec{\sigma})$, where $\vec{r}' = (2n_1(\hat{n} \cdot \vec{r}) - r_1, 2n_2(\hat{n} \cdot \vec{r}) - r_2, 2n_3(\hat{n} \cdot \vec{r}) - r_3)$ with evolution time bound $T \geq \max\left\{ \frac{\cos^{-1}(\hat{n} \cdot \vec{r})}{\sqrt{1 - (\hat{n} \cdot \vec{r})^2}}, \frac{\pi}{2} \right\} = \frac{\pi}{2}$. This shows that the inequality is indeed tight (saturated). We suppose $\hat{n} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}})$ and $\vec{r} = (0, 0, \frac{1}{2})$ as an example. Then the state $\rho(0)$ under the unitary evolution $U(T)$ becomes $\rho(T) = \frac{1}{2}(I + \vec{r}' \cdot \vec{\sigma})$, such that $\vec{r}' = (-\frac{4\sqrt{3}}{15}, \frac{\sqrt{2}}{15}, -\frac{1}{6})$. Therefore, the time bound given by Eq. (16) is approximately $\max[1.09, 0.86]$, i.e., 1.09 considering $\hbar = \omega = 1$. But a

previous bound [27, 45] would give approximately 0.31. This shows that indeed our bound is tight to the earlier ones.

In the sequel, we discuss how the geometric uncertainty relation can be measured experimentally. This is the most important implication of our new approach.

Experimental proposal.— Arguably, the most important phenomenon that lies at the heart of quantum theory is the quantum interference. It has been shown that in the interference of mixed quantum states, the visibility is given by $V = |\text{Tr}(\rho U)|$ and the relative phase shift is given by $\Phi = \text{Arg}[\text{Tr}(\rho U)]$ [51]. In quantum theory both of these play very important role and they can be measured in experiments [52, 53]. For pure quantum states, the magnitude of the visibility is the overlap of the states between the upper and lower arms of the interferometer. Therefore, for mixed states one can imagine that $|\text{Tr}(\rho U)|^2$ also represents the overlap between two unitarily connected quantum states. As defined in this paper, this visibility can be turned into a distance between ρ and $\rho' = U\rho U^\dagger$. In Fig. 1, we pass a state ρ of a system through a 50% beam splitter B1. The state in the upper arm is reflected by M and evolved by a unitary evolution operator U and the state in the lower arm is evolved by U' and then reflected through M' . Both the beams in the upper and lower arms are combined on an another 50% beam splitter B2. The beams will interfere with each other. Two detectors are placed in the receiving ends and visibility of the interference pattern is measured by counting the particle numbers received at each ends. To measure the Bargmann angle, we apply $U = U(T)$ in one arm and $U' = I$ in another arm of the interferometer. The visibility $|\text{Tr}[\rho(0)U(T)]| = \cos \frac{s_0}{2}$ will give the Bargmann angle s_0 . Once we know the visibility (the value of s_0) then we can verify the speed limit of the evolution for mixed states. To measure the quantum speed $v = \frac{2\Delta H}{\hbar}$, one can apply $U = U(t)$ in one arm of the interferometer and one applies $U' = U(t + \Delta t)$, where Δt is very small in another arm of the interferometer. Then, the visibility will be $|\text{Tr}[\rho(t)U(\Delta t)]|$. Hence, the quantum speed can be measured in terms of this visibility between two infinitesimally unitarily evolved states using the expression $V^2 = |\text{Tr}[\rho(t)U(\Delta t)]|^2 = 1 - \frac{1}{4}v^2\Delta t^2$. Thus, by appropriately changing different unitaries, we can measure the quantum speed and hence the speed limit in quantum interferometry. One can also test Levitin kind of bound using our interferometric setup. Note that R and I of Eq. (15) can be calculated from the relative phase Φ of quantum evolution together with the visibility V . The relative phase Φ of mixed state evolution can be measured by determining the shift in the interference pattern in the interferometer [52, 53]. Therefore, with prior knowledge of average of the Hamiltonian, one can test the Levitin bound for the mixed states.

This idea can be generalized also for the completely positive trace preserving (CPTP) maps. In the next section, we derive the quantum speed limit for CPTP maps.

Speed limit under CPTP map.— The metric defined in Eq. (3) gives the distance between two states which are related by unitary evolution. Now, consider a system A in a state $\rho_A(0)$

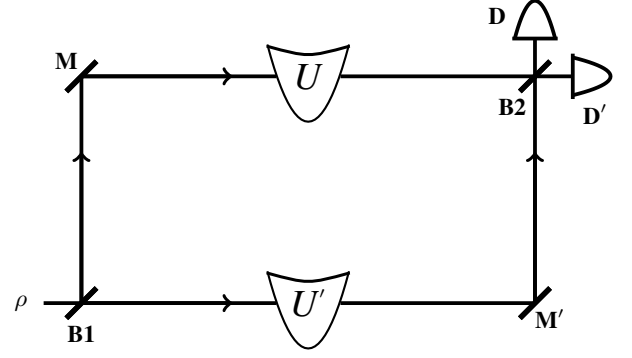


FIG. 1. Mach-Zender interferometer. An incident state ρ is beamed on a 50% beam splitter B1. The state in the upper arm is reflected through M and evolved by a unitary evolution U and the state in the lower arm is evolved by an another unitary evolution U' and then reflected through M' . Beams are combined on an another 50% beam splitter B2 and received by two detectors D and D' to measure the visibility. By appropriately choosing different unitaries, one can measure the quantum speed and the time limit.

at time $t = 0$, which evolves under CPTP map \mathcal{E} to $\rho_A(T)$ at time $t = T$. The final state $\rho_A(T)$ can be expressed in the following Kraus operator representation form as

$$\rho_A(T) = \mathcal{E}(\rho_A(0)) = \sum_k E_k(T) \rho_A(0) E_k^\dagger(T), \quad (17)$$

where $E_k(T)$'s are the Kraus operators with $\sum_k E_k^\dagger(T) E_k(T) = I$. We know that this CPTP evolution can always be represented as a unitary evolution in an extended Hilbert space via Stinespring dilation. Let us consider, without loss of generality, an initial state $\rho_{AB}(0) = \rho_A(0) \otimes |\nu\rangle_B \langle \nu|$ at time $t = 0$ in the extended Hilbert space. The combined state evolves under $U_{AB}(T)$ to a state $\rho_{AB}(T)$ such that $\rho_A(T) = \text{Tr}_B[\rho_{AB}(T)] = \mathcal{E}(\rho_A(0))$ and $E_{k=B} \langle k| U_{AB}(T) |\nu\rangle_B$ [54]. Therefore, the time required to evolve the state $\rho_A(0)$ to $\rho_A(T)$ under the CPTP evolution is the same as the time required for the state $\rho_{AB}(0)$ to evolve to the state $\rho_{AB}(T)$ under the unitary evolution $U_{AB}(T)$ in the extended Hilbert space. Following Eq. (10), we get the time bound to evolve the quantum system from $\rho_A(0)$ to $\rho_A(T)$ as

$$T \geq \frac{\hbar s_0}{2\Delta H_{AB}}, \quad (18)$$

where H_{AB} is the time independent Hamiltonian in the extended Hilbert space and $\cos \frac{s_0}{2} = |\text{Tr}[\rho_{AB}(0)U_{AB}(T)]|$. Note that the energy uncertainty of the combined system in the extended Hilbert space ΔH_{AB} can be expressed in terms of speed v of evolution of the system and the Bargmann angle s_0 can be expressed in terms of operators acting on the Hilbert space of quantum system. To achieve that, we express probability amplitude $\text{Tr}_{AB}[U_{AB}(T)(\rho_A(0) \otimes |\nu\rangle_B \langle \nu|)]$ in the extended Hilbert space in terms of linear operators acting on

the Hilbert space of quantum system as

$$\begin{aligned} & \text{Tr}_{AB}[U_{AB}(T)(\rho_A(0) \otimes |\nu\rangle_B \langle \nu|)] \\ &= \text{Tr}_A[\rho_A(0)E_\nu(T)], \end{aligned} \quad (19)$$

where $E_\nu(T) = {}_B\langle \nu | U_{AB}(T) | \nu \rangle_B$. Here, $|\text{Tr}_A[\rho_A(0)E_\nu(T)]|^2$ is the transition probability between the initial state and the final state of the quantum system under CPTP map. Therefore, we can define the Bragmann angle between $\rho_A(0)$ and $\rho_A(T)$ under the CPTP map as

$$|\text{Tr}_A[\rho_A(0)E_\nu(T)]| = \cos \frac{s_0}{2}. \quad (20)$$

Similarly, we can define the infinitesimal distance between $\rho_{AB}(0)$ and $\rho_{AB}(dt)$ connected through unitary evolution $U_{AB}(dt)$ with time independent Hamiltonian H_{AB} as

$$\begin{aligned} dD^2 &= 4(1 - |\text{Tr}[\rho_{AB}(t)U_{AB}(dt)]|^2) \\ &= 4(1 - |\text{Tr}[\rho_{AB}(0)U_{AB}(dt)]|^2) \\ &= 4(1 - |\text{Tr}[\rho_A(0)E_\nu(dt)]|^2) \\ &= \frac{4}{\hbar^2} [\text{Tr}(\rho_A(0)\tilde{H}^2_A) - [\text{Tr}(\rho_A(0)\tilde{H}_A)]^2] dt^2, \end{aligned} \quad (21)$$

where $\tilde{H}_{A=B} = {}_B\langle \nu | H_{AB} | \nu \rangle_B$ and $\tilde{H}^2_{A=B} = {}_B\langle \nu | H_{AB}^2 | \nu \rangle_B$. Therefore, the speed of the quantum system is given by

$$v^2 = \frac{4}{\hbar^2} [\text{Tr}(\rho_A(0)\tilde{H}^2_A) - [\text{Tr}(\rho_A(0)\tilde{H}_A)]^2]. \quad (22)$$

Hence, the time bound for the CPTP evolution from Eq. (18) becomes

$$T \geq \frac{s_0}{v}, \quad (23)$$

where $s_0 = 2 \cos^{-1} |\text{Tr}_A[\rho_A(0)E_\nu(T)]|$.

Here the interpretation of this limit is different from that of the unitary case. The transition probability in unitary case is symmetric with respect to the initial and the final states. Hence, the time limit can be regarded as the minimum time to evolve the initial state to the final state as well as the final state to the initial state. But the transition probability defined for positive map is not symmetric with respect to the initial and final states of the quantum system. In this case, time limit can only be regarded as the minimum time to evolve the initial state to the final state.

Since we have mapped the time bound to evolve an initial state $\rho_A(0)$ to the final state $\rho_A(T)$ under CPTP evolution with the time bound of corresponding unitary representation $\rho_{AB}(T) = U_{AB}(T)\rho_{AB}(0)U_{AB}^\dagger(T)$ of the CPTP map in the extended Hilbert space, this speed limit can be measured in the interference experiment by interfering the two states $\rho_{AB}(0)$ and $\rho_{AB}(T)$ in the extended Hilbert space.

Example of speed limit for CPTP map.— We again consider a general single qubit state $\rho_A(0) = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})$ at time $t=0$, such that $|\vec{r}|^2 \leq 1$. It evolves to $\rho_A(T)$ at time $t=T$ under completely positive trace preserving (CPTP) map $\mathcal{E} : \rho_A(0) \rightarrow \mathcal{E}(\rho_A(0)) = \rho_A(T) = \sum_k E_k(T)\rho_A(0)E_k^\dagger(T)$. This evolution is

equivalent to a unitary evolution of $\rho_{AB}(0) = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}) \otimes |0\rangle\langle 0| \rightarrow \rho_{AB}(T) = U_{AB}(T)\rho_{AB}(0)U_{AB}^\dagger(T)$ in the extended Hilbert space. The unitary evolution is implemented by a Hamiltonian $H = \sum_i \mu_i \sigma_A^i \otimes \sigma_B^i$. This is a canonical two qubit Hamiltonian up to local unitary operators. With the unitary $U_{AB}(T) = e^{\frac{iT}{\hbar} (\sum_i \mu_i \sigma_A^i \otimes \sigma_B^i)}$, we have the Kraus operators $E_0(T) = {}_B\langle 0 | U_{AB}(T) | 0 \rangle_B$ and $E_1(T) = {}_B\langle 1 | U_{AB}(T) | 0 \rangle_B$ and it is now easy to show from Eq. (23) that the time bound for this CPTP evolution is given by

$$T \geq \frac{\hbar \cos^{-1} K}{\sqrt{\mu_1^2 + \mu_2^2 + \mu_3^2(1 - r_3^2) - 2\mu_1\mu_2r_3}}, \quad (24)$$

where $K = [(\cos \theta_1 \cos \theta_2 \cos \theta_3 + r_3 \sin \theta_1 \sin \theta_2 \cos \theta_3)^2 + (\sin \theta_1 \sin \theta_2 \sin \theta_3 + r_3 \cos \theta_1 \cos \theta_2 \sin \theta_3)^2]^{\frac{1}{2}}$ and $\theta_1 = \frac{\mu_1 T}{\hbar}$, $\theta_2 = \frac{\mu_2 T}{\hbar}$ and $\theta_3 = \frac{\mu_3 T}{\hbar}$. If we consider $\theta_1 = \pi$, $\theta_3 = \pi$ then this bound reduces to $T \geq \frac{\hbar \theta_2}{\sqrt{\mu_1^2 + \mu_2^2 + \mu_3^2(1 - r_3^2) - 2\mu_1\mu_2r_3}}$. One can also check our speed bound for various CPTP maps and it is indeed respected.

Conclusion.— Quantum Interference plays a very important role in testing new ideas in quantum theory. Motivated by interferometric setup for measuring the relative phase and the visibility for the mixed states, we have proposed a new measure of distance for the mixed states, which are connected by the unitary orbit. The new metric reduces to the Fubini-Study metric for pure states. Using this metric, we have derived a geometric uncertainty relation for mixed states, which sets a quantum speed limit for arbitrary unitary evolution. In addition, a Levitin kind of bound is derived using our formalism. This new speed limit based on our formalism is tighter than any other existing bounds. Then, we have proposed an experiment to measure this new distance and quantum speed in the interference of mixed states. The visibility in quantum interference is a direct measure of distance between two mixed states of the quantum system along the unitary orbit. We have argued that by appropriately choosing different unitaries in the upper and lower arm of the interferometer one can measure the quantum speed and the Bargmann angle. This provides way to measure the quantum speed and quantum distance in quantum interferometry. We furthermore, extended the idea of speed limit for the case of completely positive trace preserving maps also. We hope that our proposed metric will lead to direct test of quantum speed limit in quantum interferometry.

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- [56] If $[\rho_1, U] = 0$, then we have $\rho_1 = \rho_2$. In this case we have either $U = I$ or U leads to a cyclic evolution. Cyclic evolution in time means there is a time period T for which we have $\rho_2 = \rho(T) = U(T)\rho_1 U^\dagger(T) = \rho_1$. In both the cases, $D(\rho_1, \rho_2) = 0$. To see the converse, i.e., if $D(\rho_1, \rho_2) = 0$, we have $[\rho_1, U] = 0$ consider the purification. From Eq. (5), considering $\rho_1 = \rho_A(t_1)$ and $\rho_2 = \rho_A(t_2)$, we have $D(\rho_1, \rho_2) = 4(1 - |\langle \Psi_{AB}(t_1) | \Psi_{AB}(t_2) \rangle|^2)$ where $|\Psi_{AB}(t_2)\rangle = U_A(t_2, t_1) \otimes I_B |\Psi_{AB}(t_1)\rangle$. In the extended Hilbert space, $D(\rho_1, \rho_2) = 0$ implies $|\langle \Psi_{AB}(t_1) | \Psi_{AB}(t_2) \rangle|^2 = 1$ and hence, $\Psi_{AB}(t_1)$ and $\Psi_{AB}(t_2)$ are same up to $U(1)$ phases. This is equivalent to $[\rho_1, U] = 0$.